MATH-F410 – REP. DES GROUPES & APP. A LA PHYS.

Solutions des exercices 2023-2024

Séance 10 : Représentations de $su(3)$ - Wigner-Eckart - Gell-Mann

1. Hadrons and $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$

Let us do it by tensorial methods. Consider three tensors in the $\mathbf{3} = (1,0)$ irrep: u^i, v^j, w^k. Let us start by taking the tensor product of the first two,

$$u^i v^j = u^{(i}v^{j)} + u^{[i}v^{j]} = u^{(i}v^{j)} + \frac{1}{2}\epsilon^{ijk}\left(\epsilon_{mnk}u^m v^n\right) \ .$$

Call $M^{ij} \equiv u^{(i}v^{j)}$ the first piece, transforming in the $(2,0) = \mathbf{6}$, and $z_k \equiv \epsilon_{mnk}u^m v^n$, transforming in the $(0,1) = \bar{\mathbf{3}}$. We can now take the tensor product with the extra vector in the $\mathbf{3}$, w^k. To avoid doing again work already done, we refer to the solutions to the previous exercise sheet, where we showed that $\mathbf{3} \otimes \mathbf{6} = \mathbf{10} \oplus \mathbf{8}$. The only new piece is then the $\mathbf{3} \otimes \bar{\mathbf{3}}$:

$$z_p w^k = \left(z_p w^k - \frac{1}{3}\delta^k_p z_q w^q\right) + \frac{1}{3}\delta^k_p z_q w^q \ .$$

The first term is traceless, therefore it transforms in the $(1,1) = \mathbf{8}$, while the second one transforms trivially (so in the $(0,0) = \mathbf{1}$). All in all, we conclude $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3} = \mathbf{10}\oplus\mathbf{8}\oplus\mathbf{8}\oplus\mathbf{1}$. We can explicitly write the full tensor decomposition as

$$u^i v^j w^k = M^{(ij}w^{k)} + \frac{2}{3}\left(\delta^{(i}_q \epsilon^{j)kp}\right)\left(\epsilon_{plm}M^{ql}v^m\right) + \frac{1}{2}\epsilon^{ijp}\left(z_p w^k - \frac{1}{3}\delta^k_p z_q w^q\right) + \frac{1}{6}\epsilon^{ijk}z_q w^q \ ,$$

with the previous definitions, $M^{ij} \equiv u^{(i}v^{j)}$ and $z_k \equiv \epsilon_{mnk}u^m v^n$, and where the first two terms come from the decomposition in the previous exercise sheet. As a bonus, you may try to verify the decomposition $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}$ by the method of Young tableaux.

2. Hadronic resonances in the $(3,0) = \mathbf{10}$

Section a)

The fastest way to obtain the weight diagram is the one described in the final exercises of the previous exercise sheet. We consider the highest-weight state, associated to the component v^{111} in the tensor analysis. The corresponding weight is $3\mu^1 = (3/2, \sqrt{3}/2)$. Each time we change $1 \to 2$ as an upper index we decrease the weight by $\alpha_1 + \alpha_2 = (1,0)$; similarly, each time we change $1 \to 3$ we decrease the weight by $\alpha_1 = (1/2, \sqrt{3}/2)$. It is almost immediate by this procedure to obtain the diagram sketched below.

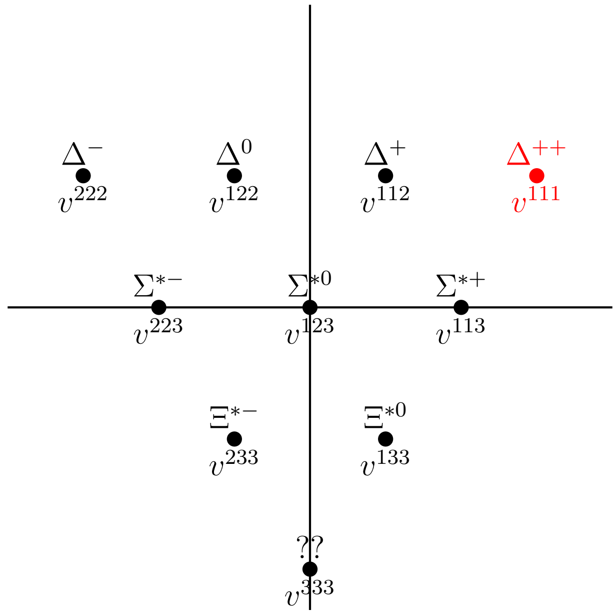

Figure 1: Weights of the representation $(3,0) = \mathbf{10}$, obtained by tensor methods as described in the main text. We label them by the non-zero component of the tensor transforming with the corresponding weight, as well as by the name of the resonances provided in Section b). In red, the highest weight of the representation.

Section b)

By the definitions in the question, the electric charge is given as $Q = I + Y/2 = h_1 + h_2/\sqrt{3}$ (in a slight abuse of notation, h_1 and h_2 refer here to the eigenvalues of the two Cartan generators). Similarly, the isospin directly gives h_1, since $I = h_1$. From the table provided in the question we can directly identify the particles, once we read h_1 and h_2 from I and Q. This is indicated in Figure 1.

Section c)

There is a weight, $(0, -\sqrt{3})$, with no assigned particle at the moment, so we can predict there should be one there. Reading the values of the isospin and electric charge from the eigenvalues of the Cartan generators, we get:

$$I = h_1 = 0 \ , \qquad Q = h_1 + \frac{h_2}{\sqrt{3}} = -1 \ .$$

This particle, the Ω^-, was indeed found a bit after Gell-Mann predicted its existence.

3. More about the Ω^-

Section a)

The isospin subalgebra is a $\mathfrak{su}(2)$ subalgebra of $\mathfrak{su}(3)$ which has as Cartan generator

$I^3 = T^3 = h_1$. Looking at the expressions for the $\mathfrak{su}(2)_\alpha$ algebra associated with any root,

$$E^3 = \frac{1}{|\alpha|^2}\alpha \cdot h \ , \qquad E^\pm = \frac{1}{|\alpha|} e_{\pm\alpha} \ ,$$

it is clear that we must take $\alpha = (1,0) = \alpha_1 + \alpha_2$ (where $\alpha_{1,2}$ are the simple roots) for the isospin subalgebra. These operators satisfy the usual $\mathfrak{su}(2)$ algebra (we denote by I^3, $I^\pm$ the generators for this specific $\alpha = (1,0)$):

$$[I^3, I^\pm] = \pm I^\pm \ , \qquad [I^+, I^-] = I^3 \ .$$

Written in terms of the $\mathfrak{su}(3)$ generators, it is easy to check in the notes that $e_{\pm(1,0)} = I^\pm = \frac{1}{\sqrt{2}}(T^1 \pm iT^2)$. Explicitly,

$$I^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ , \quad I^- = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ , \quad I^3 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ .$$

The commutation relations between these operators and $M = M_0\mathbf{1} + \mu T^8$ can be either computed explicitly with the form of the matrices or using general results. Since $h_2 = T^8$ is the second element of the Cartan subalgebra in the usual conventions, clearly $[I^3, T^8] = [h_1, h_2] = 0$, therefore $[I^3, M] = 0$. Furthermore, $[T^8, I^\pm] = [h_2, e_{\pm(1,0)}] = 0$, so we also have $[I^\pm, M] = 0$. This is the important lesson: the isospin operators commute with the mass operator proposed by Gell-Mann, therefore states within the same isospin multiplet have all the same mass. The final commutator asked in the question is $[M, T^8] = 0$, trivially.

Section b)

Due to the Wigner-Eckart theorem, matrix elements of the mass operators between states in the **10** only depend on a single reduced matrix element. This is a consequence of the decomposition of the product of the representation of the mass operator (**8**) and that of the state (**10**), done in the previous exercise sheet: $\mathbf{8} \otimes \mathbf{10} = \mathbf{8} \oplus \mathbf{10} \oplus \mathbf{27} \oplus \mathbf{35}$. There is a single factor of **10** in the decomposition contributing to the matrix element.

Section c)

Call $B(P)$ the state in the **10** representing a given particle P. Then, with the given form of the operator M, provided $B(P)$ is an eigenstate of $T^8 = h_2$:

$$MB(P) = \left(M_0 + \mu\nu^2(B(P))\right) B(P) \ ,$$

with $\nu^2(B(P))$ the second component of the weight associated to $B(P)$. The expression between parentheses is the mass of the particle P, from which we see that states with the same ν^2 have the same mass (these are the ones within isospin multiplets), while those in different rows in Figure 1 satisfy:

$$M_{\Sigma^\star} - M_\Delta = M_{\Xi^\star} - M_{\Sigma^\star} = M_\Omega - M_{\Xi^\star} \ .$$

Section d)

With the given values,

$$
\begin{aligned}
M_{\Sigma^\star} - M_\Delta &= 155\,\mathrm{MeV} \ , \\
M_{\Xi^\star} - M_{\Sigma^\star} &= 145\,\mathrm{MeV} \ , \\
M_\Omega - M_{\Xi^\star} &= ???\,\mathrm{MeV} \ ,
\end{aligned}
$$

A reasonable estimate is that the last difference is $\sim 150\,\mathrm{MeV}$, and then the mass of the Ω^- would be $M_\Omega = 1670\,\mathrm{MeV}$. The measured mass of this particle, found a bit after Gell-Mann predicted it, was $\sim 1672\,\mathrm{MeV}$.